

# A Complete Characterization of Unitary Quantum Space

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*Our motivation:* How powerful are quantum computers with a small number of qubits?

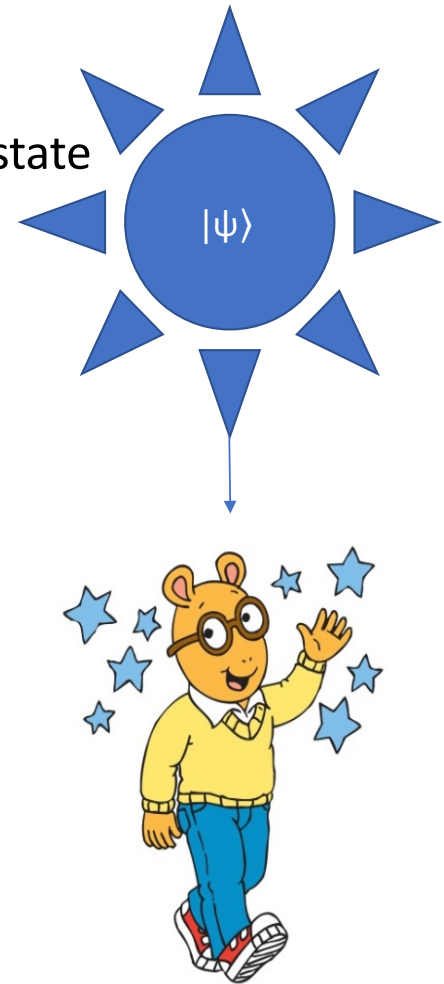
- *Our results:* Give two natural problems *characterize* the power of quantum computation with *any* bound on the number of qubits
  1. **Precise Succinct Hamiltonian** problem
  2. **Well-conditioned Matrix Inversion** problem
- These characterizations have many applications
  - **QMA** proof systems and Hamiltonian complexity
  - The power of preparing **PEPS** states vs ground states of **Local Hamiltonians**
  - Classical **Logspace** complexity

# Quantum space complexity

- **BQSPACE**[ $k(n)$ ] is the class of promise problems  $L=(L_{yes},L_{no})$  that can be decided by a bounded error quantum algorithm acting on  $k(n)$  qubits.
  - i.e., Exists uniformly generated family of quantum circuits  $\{Q_x\}_{x \in \{0,1\}^*}$  each acting on  $O(k(|x|))$  qubits:
    - “If answer is yes, the circuit  $Q_x$  accepts with high probability”
$$x \in L_{yes} \Rightarrow \langle 0^k | Q_x^\dagger | 1 \rangle \langle 1 |_{out} Q_x | 0^k \rangle \geq 2/3$$
    - “If answer is no, the circuit  $Q_x$  accepts with low probability”
$$x \in L_{no} \Rightarrow \langle 0^k | Q_x^\dagger | 1 \rangle \langle 1 |_{out} Q_x | 0^k \rangle \leq 1/3$$
- Our results show two natural complete problems for **BQSPACE**[ $k(n)$ ]
  - For any  $k(n)$  so that  $\log(n) \leq k(n) \leq \text{poly}(n)$
  - Our reductions use classical  $k(n)$  space and  $\text{poly}(n)$  time
- *Subtlety*: This is “unitary quantum space”
  - No intermediate measurements
  - Not known if “deferring” intermediate measurements can be done space efficiently

# Quantum Merlin-Arthur

- Problems whose solutions can be verified quantumly given a quantum state as witness
- **QMA**( $c,s$ ) is the class of promise problems  $L=(L_{yes},L_{no})$  so that:
  - $x \in L_{yes} \Rightarrow \exists |\psi\rangle \Pr[V(x, |\psi\rangle) = 1] \geq c$
  - $x \in L_{no} \Rightarrow \forall |\psi\rangle \Pr[V(x, |\psi\rangle) = 1] \leq s$
- **QMA** = **QMA**( $2/3,1/3$ ) =  $\bigcup_{c>0}$  **QMA**( $c,c-1/\text{poly}$ )
- $k$ -Local Hamiltonian problem is **QMA**-complete (when  $k \geq 2$ ) [Kitaev '00]
  - Input:  $H = \sum_{i=1}^M H_i$ , each term  $H_i$  is  $k$ -local
  - Promise either:
    - Minimum eigenvalue  $\lambda_{\min}(H) > b$  or  $\lambda_{\min}(H) < a$
    - Where  $b-a \geq 1/\text{poly}(n)$
  - Which is the case?
- Generalizations of **QMA**:
  1. **PreciseQMA** =  $\bigcup_{c>0}$  **QMA**( $c,c-1/\text{exp}$ )
  2. **k-bounded QMA** $_m(c,s)$ 
    - Arthur's verification circuit acts on  $k$  qubits
    - Merlin sends an  $m$  qubit witness



Characterization 1:  
**Precise Succinct Hamiltonian** problem

# The *Precise Succinct* Hamiltonian Problem

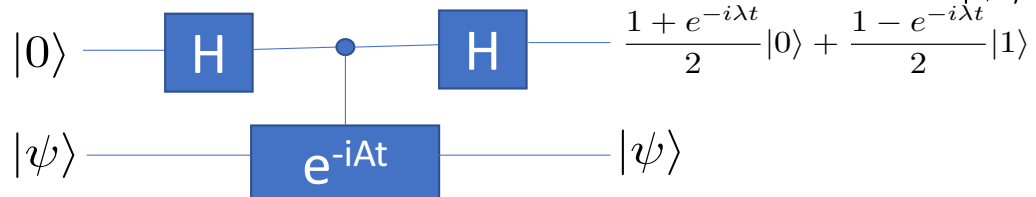
- Definition: “*Succinct Encoding*”
  - We say a classical Turing machine  $M$  is a *Succinct Encoding* for  $2^{k(n)} \times 2^{k(n)}$  matrix  $A$  if:
    - On input  $i \in \{0,1\}^{k(n)}$ ,  $M$  outputs non-zero elements in  $i$ -th row of  $A$
    - Using at most  $\text{poly}(n)$  time and  $k(n)$  space
- $k(n)$ -**Precise Succinct Hamiltonian** problem
  - Input: Size  $n$  Succinct Encoding of  $2^{k(n)} \times 2^{k(n)}$  Hermitian PSD matrix  $A$
  - Promised either:
    - Minimum eigenvalue  $\lambda_{\min}(A) > b$  or  $\lambda_{\min}(A) < a$
    - Where  $b-a > 2^{-O(k(n))}$
  - Which is the case?
- Compared to the **Local Hamiltonian** problem...
  - Input is Succinctly Encoded instead of Local
  - Precision needed to determine the promise is  $1/2^k$  instead of  $1/\text{poly}(n)$
- *Our Result*:  $k(n)$ -**P.S Hamiltonian** problem is *complete* for **BQSPACE** $[k(n)]$

Upper bound (1/2):

$k(n)$ -P.S Ham.  $\in k(n)$ -bounded  $\text{QMA}_{k(n)}(c, c-2^{-k(n)})$

- *Recall:*  $k(n)$ -Precise Succinct Hamiltonian problem
  - Given Succinct Encoding of  $2^{k(n)} \times 2^{k(n)}$  Hermitian PSD matrix  $A$ , is  $\lambda_{\min}(A) \leq a$  or  $\lambda_{\min}(A) \geq b$  where  $b-a \geq 2^{-O(k(n))}$ ?

- Merlin send eigenstate  $|\psi\rangle$  with minimum eigenvalue
  - Arthur runs phase estimation with one ancilla qubit on  $e^{-iA}$  and  $|\psi\rangle$



- Measure ancilla and accept iff “0”
- Easy to see that we get “0” outcome with probability that’s slightly ( $2^{-O(k)}$ ) higher if  $\lambda_{\min}(A) < a$  than if  $\lambda_{\min}(A) > b$
- But this is exactly what’s needed to establish the claimed bound!
- *Remaining question:* how do we implement  $e^{-iA}$ ?
  - We need to implement this operator with precision  $2^{-k}$ , since otherwise the error in simulation overwhelms the gap!
  - Luckily, we can invoke recent “precise Hamiltonian simulation” results of [Childs et. al’14]
    - Implement  $e^{-iA}$  to within precision  $\epsilon$  in space that scales with  $\log(1/\epsilon)$  and time  $\text{polylog}(1/\epsilon)$
    - See also Guang Hao Low’s talk on Thursday!
- Using these results, can implement Arthur’s circuit in  $\text{poly}(n)$  time and  $O(k(n))$  space

Upper bound (2/2):

$$k(n)\text{-bounded QMA}_{k(n)}(c, c-2^{-k(n)}) \subseteq \text{BQSPACE}[k(n)]$$

1. Error amplify the **PreciseQMA** protocol
    - *Goal*: Obtain a protocol with error inverse exponential in the witness length,  $k(n)$
    - We want to do this while simultaneously preserving verifier space  $O(k(n))$
    - We develop new “space-preserving” **QMA** amplification procedures
      - By combining ideas from “in-place” amplification [Marriott & Watrous ‘04] with phase estimation
  2. “Guess the witness”!
    - Consider this amplified verification protocol run on a maximally mixed state on  $k(n)$  qubits
    - Not hard to see that this new “no witness” protocol has a “precise” gap of  $O(2^{-k(n)})!$
  3. Amplify again!
    - Use our “space-efficient” **QMA** error amplification technique again!
    - Obtain bounded error, at a cost of exponential time
    - But the space remains  $O(k(n))$ , establishing the **BQSPACE** $[k(n)]$  upper bound
- Space-efficient amplification also used to prove hardness!
    - $k(n)$ -P.S Hamiltonian is **BQSPACE** $[k(n)]$ -hard
    - Follows from first using our space-bounded amplification, and then Kitaev’s clock-construction to build sparse Hamiltonian from the amplified circuit



## Application: **PreciseQMA=PSPACE**

- *Question:* How does the power of **QMA** scale with the completeness-soundness gap?
- *Recall:* **PreciseQMA** =  $\bigcup_{c>0} \text{QMA}(c, c-2^{-\text{poly}(n)})$
- Both upper and lower bounds follow from our completeness result, together with **BQPSPACE=PSPACE** [Watrous'03]
- *Corollary:* “**precise k-Local Hamiltonian problem**” is **PSPACE**-complete
- *Extension:* “Perfect Completeness case”: **QMA(1, 1-2<sup>-poly(n)</sup>)=PSPACE**
  - *Corollary:* checking if a local Hamiltonian has zero ground state energy is **PSPACE**-complete

# Where is this power coming from?

- Could **QMA=PreciseQMA=PSPACE**?
  - Unlikely since **QMA=PreciseQMA  $\Rightarrow$  PSPACE=PP**
    - Using **QMA  $\subseteq$  PP**
- How powerful is **PreciseMA**, the *classical analogue* of **PreciseQMA**?
  - *Crude upper bound*: **PreciseMA  $\subseteq$  NP<sup>PP</sup>  $\subseteq$  PSPACE**
  - And believed to be strictly less powerful, unless the “Counting Hierarchy” collapses
- So the power of **PreciseQMA** seems to come from both the quantum witness and the small gap, together!

# Understanding “Precise” complexity classes

- We can answer questions in the “precise” regime that we have no idea how to answer in the “bounded-error” regime
- *Example 1*: How powerful is **QMA(2)**?
  - **PreciseQMA=PSPACE** (our result)
  - **PreciseQMA(2)=NEXP** [Blier & Tapp’07, Pereszlényi’12]
  - So, **PreciseQMA(2) ≠ PreciseQMA**, unless **NEXP=PSPACE**
- *Example 2*: How powerful are quantum vs classical witnesses?
  - **PreciseQCMA**  $\subseteq$  **NP<sup>PP</sup>**
  - So, **PreciseQMA**  $\neq$  **PreciseQCMA**, unless **PSPACE**  $\subseteq$  **NP<sup>PP</sup>**
- *Example 3*: How powerful is **QMA** with perfect completeness?
  - **PreciseQMA=PreciseQMA<sub>1</sub>=PSPACE**

# Characterization 2: **Well-Conditioned Matrix Inversion**

# The Classical Complexity of Matrix Inversion

- The **Matrix Inversion** problem

- Input: nonsingular  $n \times n$  matrix  $A$  with integer entries, promised either:

- $A^{-1}[0,0] > 2/3$  or

- $A^{-1}[0,0] < 1/3$

- Which is the case?

$$\mathbf{A} = \begin{pmatrix} a_{0,0} & a_{0,1} \dots \\ \vdots & \\ a_{n,0} & a_{n,1} \dots \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} ? \dots ? \\ \vdots \\ ? \dots ? \end{pmatrix}$$

- This problem can be solved in classical  $O(\log^2(n))$  space [Csanky'76]

- Not believed to be solvable classically in  $O(\log(n))$  space

- If it is, then  $\mathbf{L}=\mathbf{NL}$  (**Logspace** equivalent of  $\mathbf{P}=\mathbf{NP}$ )

# Can we do better quantumly?

- “**Well-Conditioned Matrix Inversion**” can be solved in *non-unitary* **BQSPACE** $[\log(n)]$ ! [Ta-Shma’12] building on [HHL’08]
  - i.e., same problem with  $\text{poly}(n)$  upper bound on the condition number,  $\kappa$ , so that  $\kappa^{-1}I < A < I$
  - *Appears* to attain quadratic speedup in space usage over classical algorithms
- *Begs the question*: how important is this “well-conditioned” restriction?
  - Can we also solve the *general* **Matrix Inversion** problem in quantum space  $O(\log(n))$ ?

# Our results on Matrix Inversion

- **Well-conditioned Matrix Inversion** is complete for *unitary* **BQSPACE**[log(n)]!
  1. We give a new quantum algorithm for **Well-conditioned Matrix Inversion** avoiding intermediate measurements
    - Combines techniques from [HHL'08] with amplitude amplification
  2. We also prove **BQSPACE**[log(n)] hardness— suggesting that “well-conditioned” constraint is *necessary* for quantum **Logspace** algorithms

Can generalize from  $\log(n)$  to  $k(n)$  qubits...

- **Result 3:  $k(n)$ -Well-conditioned Matrix Inversion is complete for  $\text{BQSPACE}[k(n)]$** 
  - Input: Succinct Encoding of  $2^k \times 2^k$  PSD matrix  $A$ 
    - Upper bound  $\kappa < 2^{O(k(n))}$  on the condition number so that  $\kappa^{-1}I < A < I$
  - Promised either  $|A^{-1}[0,0]| \geq 2/3$  or  $\leq 1/3$
  - Decide which is the case?
- Additionally, by varying the dimension and the bound on the condition number, can use **Matrix Inversion** problem to *characterize* the power of quantum computation with simultaneously bounded time *and* space!



# Open questions

- Can we use our **PreciseQMA=PSPACE** characterization to give a **PSPACE** upper bound for other complexity classes?
  - For example, **QMA(2)**?
- How powerful is **PreciseQIP**?
- Natural complete problems for *non-unitary* quantum space?

Thanks!