The Power of Quantum Fourier Sampling

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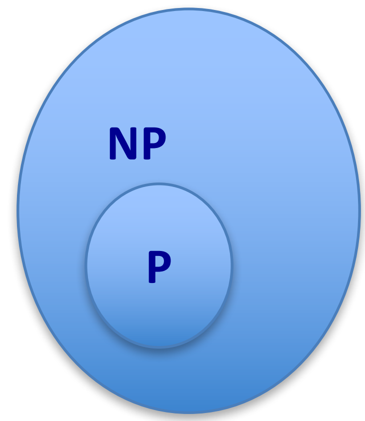
I. Complexity Theory Basics

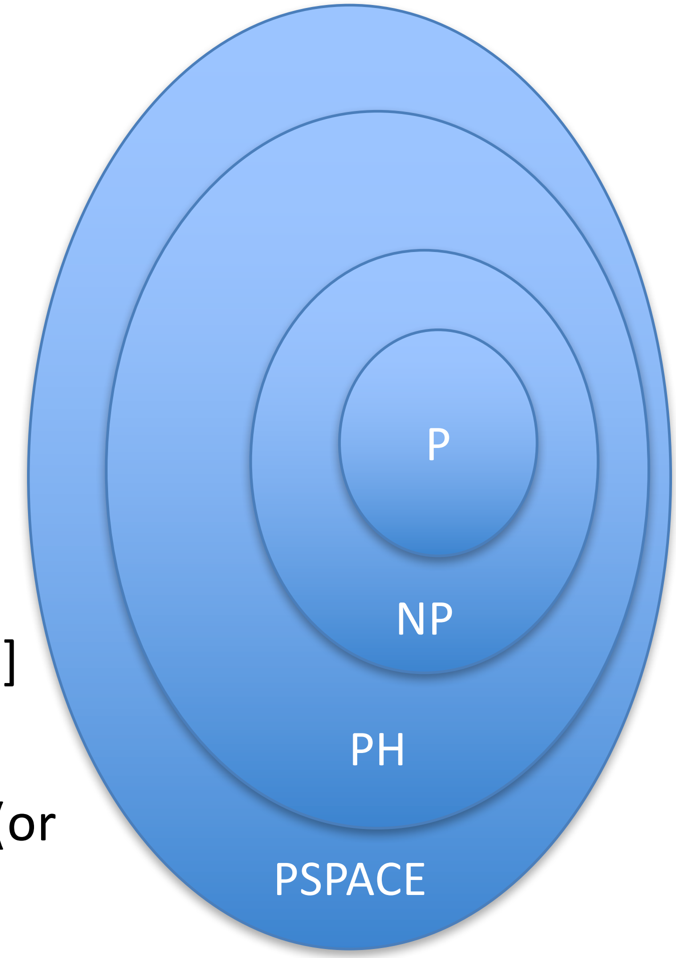
Classical Complexity Theory

- **P**
  - Class of problems efficiently solved on classical computer
- **NP**
  - Class of problems with efficiently checkable solutions
  - Characterized by **SAT**
    - Input: $\Psi:\{0,1\}^n \rightarrow \{0,1\}$
      - n-variable boolean formula
        - » E.g., $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee -x_2 \vee x_6) \wedge \ldots$
    - Problem: $\exists\, x_1, x_2, \ldots, x_n$ so that $\Psi(x)=1$?
  - Could use a box solving **SAT** to solve any problem in **NP**



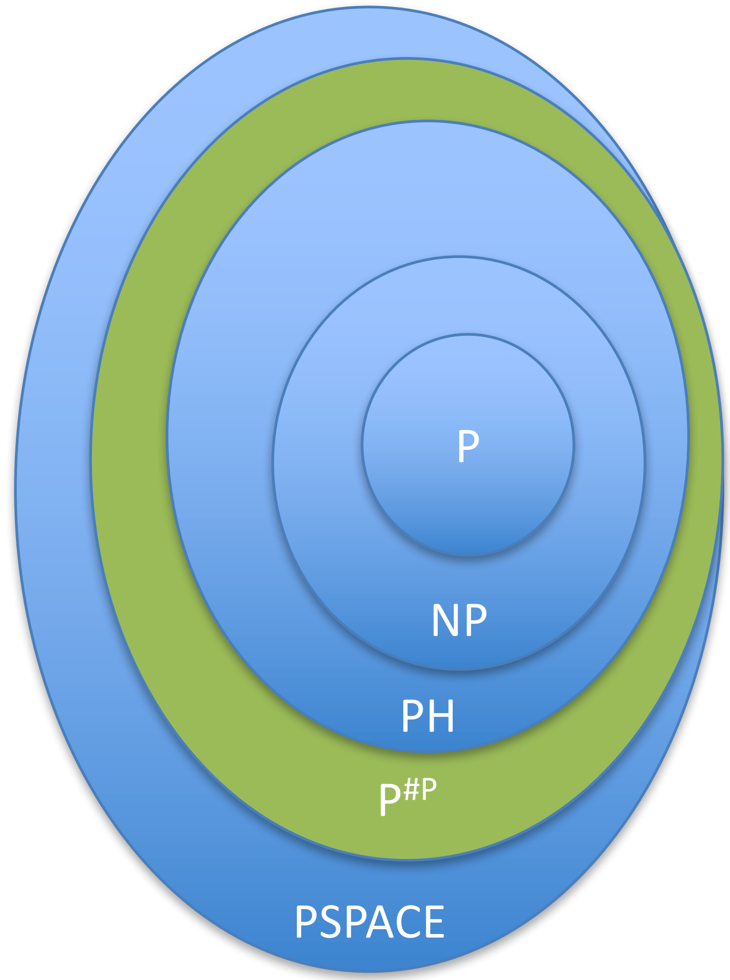
Beyond NP

- **Tautology**
  - Input: $\Psi:\{0,1\}^n \rightarrow \{0,1\}$
  - $\forall x\ \Psi(x)=1$?
  - Complete for **coNP**
- **QSAT$_k$**
  - Generalizes **SAT** and **Tautology**
  - Input: $\Psi:\{0,1\}^n \rightarrow \{0,1\}$ & partitioning $S_1,S_2,\ldots,S_k \subseteq [n]$
    - Problem: $\exists x_{S1} \forall x_{S2},\ldots,Q_k x_{Sk}$ so that $\Psi(x)=1$?
      - Thought to be strictly harder with larger k's (or else there is a collapse)
- **Σ_k** is class of problems solvable with a **QSAT$_k$** box
- **PH** is class of problems solvable with a **QSAT$_{O(1)}$** box
- **PSPACE** is class of problems solvable with a **QSAT$_n$** box



Complexity of *Counting*

- **#SAT**
  - Input: $\Psi:\{0,1\}^n \rightarrow \{0,1\}$
  - Problem: How many satisfying assignments to Ψ?
- **#SAT** is complete for **#P**
- **PH** $\subseteq$ **P$^{\#P}$** [Toda'91]
- $\textbf{Permanent}[X] = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}$ is **#P**-hard



Complexity of *Approximate* Counting

- Given efficiently computable $f:\{0,1\}^n \rightarrow \{0,1\}$ and $y \in \{0,1\}$
  - Want to compute $\Pr_{x\sim U}[f(x)=y]$ exactly
  - This is **#P**-hard
    - Because $\Pr_x[f(x)=1]=\{\#\ x\text{'s so that } f(x)=1\}/2^n=\sum_x f(x)/2^n$
    - This is as hard as counting number of satisfying assignments to formula Ψ
- However, *estimating* $\Pr_{x\sim U}[f(x)=y]$ to within multiplicative error can be done in $\mathbf{\Sigma_3}$, the third level of **PH** [Stockmeyer '83]
  - So for input $f:\{0,1\}^n \rightarrow \{0,1\}$ and $\varepsilon>0$ can output α:

$$(1-\epsilon)\sum_x f(x) \leq \alpha \leq (1+\epsilon)\sum_x f(x)$$

  in randomized time poly(n,1/ε) with **NP** oracle
- But, situation is very different for $g:\{0,1\}^n \rightarrow \{+1,-1\}$
  - Computing $\Sigma_x g(x)$ exactly is still **#P**-hard
  - Estimating $\Sigma_x g(x)$ to within $(1\pm\varepsilon)$ *multiplicative* error is **#P**-hard!
    - Binary search & Padding
  - Can generalize this hardness:
    - Estimating $(\Sigma_x g(x))^2$ to within $(1\pm\varepsilon)$ *multiplicative* error is **#P**-hard
  - Why is this so much harder than the {0,1}-valued case?
    - Cancellations

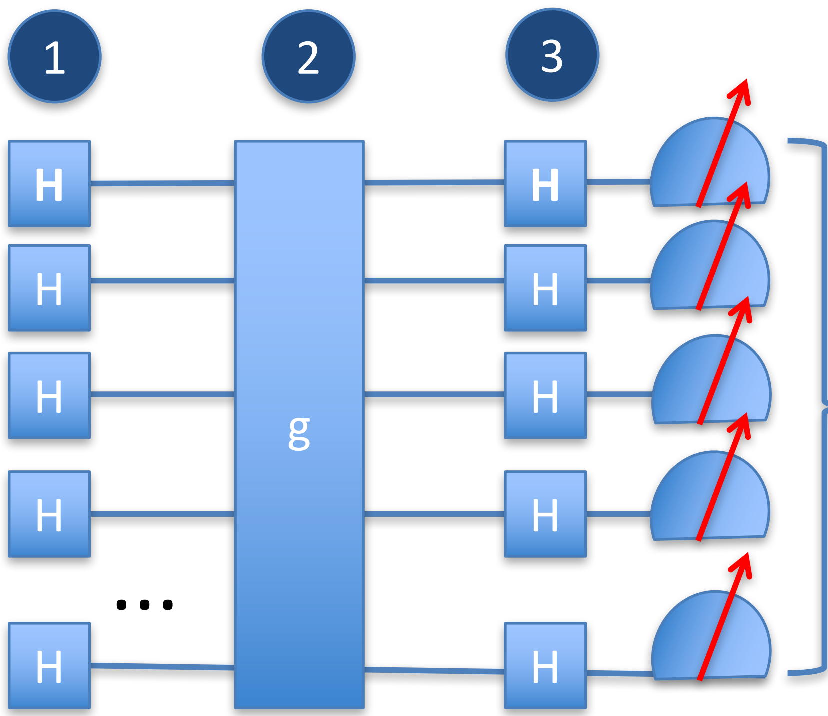
Today

- Want to show that quantum computers are capable of sampling from distributions that cannot be sampled by randomized classical algorithms
- Two constructions of hard distributions
    1. "Exact" construction
        - No classical algorithm can sample from exactly the same distribution as the quantum algorithm
    2. "Approximate" construction
        - *Goal*: Show no classical algorithm can sample from any distribution even close (in total variation distance) to quantum distribution
        - Why do we want to do this?
            - "To model error"
            - [Aaronson '11] has shown that such a result would imply a "function problem" complexity separation (i.e., **fBQP**$\not\subset$ **fBPP**)…
    - *Upshot*: We'll reach many of the same conclusions of the *BosonSampling* [AA'10] proposal with a (conceptually) much simpler setup. Our proposal also weakens the hardness conjectures needed by [AA'10], but as of yet does not resolve them….

II. “Exact” Construction [implicit in *Aaronson ‘11*]

Quantumly sampleable distribution

- *Recall*: For efficiently computable function g:$\{0,1\}^n \rightarrow \{\pm 1\}$, giving a $(1 \pm \varepsilon)$ mult error estimate to $(\sum_x g(x))^2$ is **#P**-hard
- Consider the following quantum circuit:



Distribution over $\{0,1\}^n$

1 $$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

2 $$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} g(x)|x\rangle$$

3 $$\frac{1}{2^n} \sum_{y \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} -1^{\langle x,y \rangle} g(x)|y\rangle$$

Key point: The probability of seeing 00...0 is $(\sum_x g(x))^2/2^{2n}$

Exact classical sampler collapses **PH**

- Suppose C is a randomized algorithm that samples the outcome distribution so by definition:

$$\Pr_{r\sim U_{p(n)}}[C(r)=y]=\frac{1}{2^{2n}}\left(\sum_{x\in\{0,1\}^n}-1^{\langle x,y\rangle}g(x)\right)^2$$

- Note that **p**=$\Pr_r[C(r)=00...0]=(\sum_x g(x))^2/2^{2n}$ encodes a **#P**-hard quantity
- Use Stockmeyer's algorithm to find a (1±ε) multiplicative error estimate to **p**
- Puts $\mathbf{P^{\#P}} \subseteq \mathbf{\Sigma_3}$ (but Toda tells us that $\mathbf{PH} \subseteq \mathbf{P^{\#P}}$)
- $\mathbf{PH} \subseteq \mathbf{\Sigma_3}$ (collapse!!)

How *robust* is this prior construction?

- Not very!!
  - Hardness based on a single exp. small probability
  - *Definition*: For distribution X over $\{0,1\}^n$:

Given as input ε>0, suppose a classical randomized algorithm samples from any distribution Y, with $|X\text{-}Y|_1<\varepsilon$, in time poly(n,1/ε)

Call such a classical algorithm an "*Approximate Sampler*" for X

- **Our goal**: Find a quantumly sampleable X, where the existence of a classical "Approximate Sampler" would cause **PH** collapse.
- Prior construction doesn't work! (Adversary just "erases" probability we care about)

III. “Approximate” Construction using Quantum Fourier Sampling [F., Umans ‘15]

Construction of distribution D_{PER}

- Define an efficiently computable function $h:[n!] \rightarrow \{0,1\}^{n\wedge 2}$
  - Takes a permutation in S_n to its trivial encoding as an n x n permutation matrix
  - Can be computed efficiently using e.g., Lehmer codes
  - Note h is 1-to-1 and h^{-1} also efficiently computable
- Quantum sampler:
  - Two steps:
  1. Prepare uniform superposition over n x n permutation matrices
     - Prepare uniform superposition over S_n
     - Apply h, followed by h^{-1}
  2. Hit with Hadamard on each of n^2 qubits
- Measure in standard basis

1

$$\frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\sigma\rangle |00\ldots0\rangle$$

$$\frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\sigma\rangle |h(\sigma)\rangle$$

$$\frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\sigma \oplus h^{-1}(h(\sigma))\rangle |h(\sigma)\rangle$$

$$\frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |h(\sigma)\rangle$$

2

$$\frac{1}{\sqrt{n! 2^{n^2}}} \sum_{w \in \{0,1\}^{n^2}} \underbrace{\sum_{\sigma \in S_n} (-1)^{<w,h(\sigma)>}}\, |w\rangle$$

This is the permanent of $\{\pm 1\}^{n \times n}$ matrix encoded by the string w

What's happening?

- Recall, **Permanent**($x_1,x_2,\ldots,x_{n^\wedge 2}$) is a multilinear polynomial of degree n
- Our quantum sampling algorithm (*omitting normalization*):

All possible multilinear monomials over n^2 variables $M_1,\ldots M_{2^\wedge\{n^\wedge 2\}}$

All possible matrices $X_1,X_2,\ldots,X_{2^\wedge\{n^\wedge 2\}} \in \{\pm 1\}^{n^\wedge 2}$

$$\begin{bmatrix} M_1(X_1), M_2(X_1), \ldots, M_{2^\wedge\{n^\wedge 2\}}(X_1) \\ \\ H^{\otimes n^2} \\ \\ M_1(X_{2^\wedge\{n^\wedge 2\}}), \ldots, M_{2^\wedge\{n^\wedge 2\}}(X_{2^\wedge\{n^\wedge 2\}}) \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \\ 1 \\ 0 \\ \ldots \end{pmatrix} = \begin{pmatrix} \mathbf{Per}[X_1] \\ \mathbf{Per}[X_2] \\ \\ \ldots \\ \\ \mathbf{Per}[X_{2^\wedge\{n^\wedge 2\}}] \end{pmatrix}$$

This is supported on the monomials in the **Permanent**

Approximate sampler consequences

This is extremely similar to the hardness consequence for Aaronson and Arkhipov: except their matrix distribution is iid Gaussian, $\mathcal{N}(0,1)$

- Each X ... to D_{PER}:

$$\frac{\mathrm{Per}^2[X]}{2^{n^2} n!}$$

- Suppose we had an approximate sampler, A, for D_{PER}

But our "quantum sampler" is completely different!

  - Un... ability
  - But, if A samples from distribution $\varepsilon\delta$-far from D_{PER} we know:
    - "Most probabilities in A's distribution must be close to probabilities in D_{PER}"
    - At least $(1-\delta)$-fraction of probabilities must be within $\varepsilon/2^{n^2}$ of true probability
  - *Strategy*: Choose a $X \in \{-1,+1\}^{n^2}$ matrix with iid uniformly distributed entries and approximate its probability using Stockmeyer's algorithm
- We'd obtain solution that "solves $\mathrm{Per}^2(X)$" in $\mathbf{\Sigma_3}$ with two major caveats:
  - Only "works" with probability $1-\delta$ over choice of matrix
  - "Works" means approximating within additive error $\pm\varepsilon n!$
  - **Our question**: How hard is this?
    - If it's **#P**-hard, by Toda's theorem, an approximate sampler for D_{PER} would imply a **PH** collapse (as in the exact case)

Relating Additive to Multiplicative error

- Our procedure computes:
  - $Per^2[X] \pm \epsilon n!$ with probability $1-\delta$ in Σ_3-time $poly(n,1/\epsilon,1/\delta)$ time
- This is unnatural! Would like multiplicative error:
  - $(1-\epsilon)Per^2[X] \leq \alpha \leq (1+\epsilon)Per^2[X]$ with probability $1-\delta$ in Σ_3-time $poly(n,1/\epsilon,1/\delta)$ time
- Can we get *multiplicative* error using our procedure?
  - "Permanent Anti-concentration conjecture" [AA'11]
    - Need: exists polynomial p so that for all n and δ
      - $Pr_X[|Per(X)|<\sqrt{(n!)}/p(n,1/\delta)]<\delta$
    - This may actually be true!!
    - For Bernoulli distributed $\{-1,+1\}^{n \times n}$ matrices:
      - $\forall \varepsilon>0$ $Pr_X[|Per[X]|^2<n!/n^{\varepsilon n}]<1/n^{0.1}$ [Tao & Vu '08]

How hard is "Approximating" the Permanent?

- *Scenario* 1:
  - Suppose I had a box that:
    - "Solves all the Permanents approximately"
    - Input: ϵ>0 and matrix X$\in${-1,+1}$^{n \times n}$
    - Output: α so that:
    $$(1-\epsilon)\mathsf{Per}^2(\mathsf{X}) \leq \alpha \leq (1+\epsilon)\mathsf{Per}^2(\mathsf{X})$$
    - In time poly(n,1/ϵ)
  - This is **#P**-hard!
    - Proof: "Padding and binary search!"
- *Scenario* 2:
  - Suppose I had a box that:
    - "Solves most of the Permanents exactly"
    $$\Pr_X[\alpha = \mathsf{Per}^2[\mathsf{X}]] > 1-\delta$$
    - For δ=1/poly(n)
  - This is **#P**-hard!
    - Proof idea: Polynomial interpolation [Lipton '89 in finite field case…]!
- Our "solution" has weakness of both Scenario 1 and 2
  - Hardness proofs break-down!
  - This is exactly the same reason other two "approximate" sampling results need conjectures…

Generalizations

- Entries of Matrix
  - Replace Quantum Fourier Transform over $Z_2^{n^2}$ with Quantum Fourier Transform over $Z_k^{n^2}$
    - Resulting amplitudes proportional to Permanents of matrices with entries of evenly-spaced points around unit circle
- Generalizing the distribution over matrices
  - Can recapture the Gaussian distributed entries of [AA'11]…
- "Hard Polynomial"
  - Generalize Permanent to any *Efficiently Specifiable* polynomial sampling
    - Multilinear, homogenous polynomials with *m* monomials of the form:

$$Q(X_1, X_2..., X_n) = \sum_{y \in [m]} {X_1}^{h(y)_1} {X_2}^{h(y)_2} ... {X_n}^{h(y)_n}$$

    - Where h is efficiently computable map (and h^{-1} is also)
  - Examples:
    - Permanent, Hamiltonian Cycle polynomial, many more…

Relation to other work

- There are lots of "exact" sampling results
  - Starting with [DiVincenzo-Terhal'02] and [Bremner-Jozsa-Shepherd'10]
  - These distributions can often be sampled by restrictive classes of quantum samplers
    - Constant depth quantum circuits [DT'02]
    - Quantum computations with commuting gates [BJS'10]
    - One clean qubit [Morimae et. al. 2014]
    - Etc…
- "Approximate" sampling is far less understood...
  - "Boson Sampling" [Aaronson and Arkhipov '11]
  - "IQP Sampling" [Bremner, Montanaro and Shepherd'15]
  - Quantum Fourier Sampling [F.,Umans '15]
- All rely on similar non-standard hardness assumptions
  - Need to conjecture that computing "average-case approximate" solution to some polynomial is hard for the **PH**
    - Permanent [AA'11]
    - The partition function of a random instance of an Ising model [BMS'15]
    - Any *Efficiently Specifiable* polynomial [F., Umans '15]

Another recent result related to "Semi-Quantum Computing"

- How powerful is restricted space quantum computation?
  - i.e., Quantum computation with restriction of number of qubits, but no restriction on time
- [F., Lin '16] Tight connection between Matrix inversion problem and unitary space complexity
  - $k(n)$-Matrix inversion problem
    - Given circuit "encodes" $2^{k(n)}$x$2^{k(n)}$ PSD matrix A
      - Input: row index
      - Output: non-zero elements of row
    - Upper bound on condition number $\kappa<2^{k(n)}$ so that $\kappa^{-1}I \preceq A \preceq I$
    - Promised either $|A^{-1}(s,t)|\geq b$ or $\leq a$ where a,b are constants between 0 and 1
    - Decide which is the case?
  - Complete for *unitary* **BQSPACE**[$k(n)$]
    1. Matrix inversion algorithm doesn't need intermediate measurements
    2. We also have hardness!

Thanks!